(ii) $x^2 f(t)u = e^u$

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Marking scheme Calculus I Oct 2015

Ex 1

a) Principle of Mathematical Induction

To prove a statement $B(n)$

1pt about a natural number n

1pt $B(n)$ is true for all n 1pt if $I'y = I(-\frac{2}{x})y$

1pt $B(1)$ is true

1pt $B(n)$ is true $\Rightarrow$ $B(n+1)$ is true

b) Using Mathematical induction

first step:

1pt $(f_1)' = f_1'$ or $(f_1+f_2)' = f_1' + f_2'$

induction hypothesis

1pt $(f_1+f_2+\dots+f_n)' = f_1' + f_2' + \dots + f_n'$

to be shown

1pt $(f_1+f_2+\dots+f_{n+1})' = f_1' + f_2' + \dots + f_{n+1}'$

prove

define $f = f_1+f_2+\dots+f_n$

1pt $f_1+f_2+\dots+f_n+f_{n+1} = f + f_{n+1}$

sum rule differentiation

1pt $(f+f_{n+1})' = f' + f_{n+1}'$

use induction hypothesis

$f' = f_1' + f_2' + \dots + f_n'$

and combine results.

and combine results :

1 pt $\quad (f_1 + \cdots + f_{n+1})' = f_1' + f_2' + \cdots + f_{n+1}'$

Ex 2 $\quad$ Solve $\quad e^{-iz} = \dfrac{i+1}{i-1}$

1 pt $\quad z = x + iy$

1 pt $\quad e^{-iz} = e^{y} \cdot e^{-ix}$

1 pt $\quad$ idea polar representation

1 pt $\quad i+1 = \sqrt{2}\, e^{i\left(\frac{\pi}{4} + 2k\pi\right)}$

1 pt $\quad i-1 = \sqrt{2}\, e^{i\left(\frac{3\pi}{4} + 2k\pi\right)}$

2 pt $\quad \dfrac{i+1}{i-1} = e^{i\left(-\frac{\pi}{2} + 2k\pi\right)} \qquad k \in \mathbb{Z}$

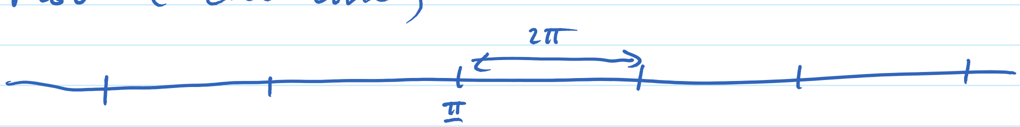
Hence :

1 pt $\quad e^{y} = 1 \implies y = 0$

1 pt $\quad -x = -\frac{\pi}{2} + 2k\pi \implies x = \frac{\pi}{2} - 2k\pi$

Plot (real line)

1 pt



Ex 3 $\quad$ a) Precise definition $\lim_{x \to a} f(x) = L$:

4 pt $\quad \underbrace{\forall \varepsilon > 0}_{1\text{ pt}} \ \underbrace{\exists \delta > 0}_{1\text{ pt}} : \underbrace{0 < |x - a| < \delta}_{1\text{ pt}} \implies \underbrace{|f(x) - L| < \varepsilon}_{1\text{ pt}}$

b) Prove $\lim x^2 =$

1pt 1pt 1pt 1pt

b) Prove $\lim_{x \to 3} x^2 = 9$

1pt $|x^2 - 9| = |x-3|\,|x+3|$

1pt $|x-3| < \delta$

1pt Take $\delta = 1$, then

$2 < x < 4$ hence

1pt $5 < |x+3| < 7$

So $|x^2 - 9| < 7\delta$

1pt if $|x-3| < \delta$ and $|x-3| < 1$

Hence according to the formal limit definition we can take

1pt $\delta = \min\{1, \frac{\varepsilon}{7}\}$

Ex 4 a) Mean Value Theorem

If the function f is

1pt · continuous on $[a,b]$

1pt · differentiable on (a,b)

then

2pt $\underbrace{\exists\, c \in (a,b)}_{1pt}$; $\overbrace{f'(c) = \frac{f(b) - f(a)}{b - a}}^{1pt}$

b) Show that $|\sin a - \sin b| \leq |a - b|$

1pt Take $f(x) = \sin x$

1pt $f'(x) = \cos x$

$-1 \leq f'(x) \leq 1$

1pt Hence $\forall c \in (a,b)$: $|f'(c)| \leq 1$

$-1 \le f'(x) \le 1$

1 pt Hence $\forall c \in (a,b)$: $|f'(c)| \le 1$

Use MVT : $\exists c \in (a,b)$

1 pt $\sin b - \sin a = \cos(c)(b-a)$

1 pt $|\sin b - \sin a| = |\cos(c)|\,|b-a|$

1 pt $\le |b-a|$

Ex 5

a) Use definition derivative to compute $f'(0)$

1 pt Definition $f'(0) = \lim_{h \to 0} \dfrac{f(h) - f(0)}{h}$

Substitutions:

1 pt $f'(0) = \lim_{h \to 0} \dfrac{e^{-1/h^2}}{h}$

Indeterminate form "$\frac{0}{0}$"

1 pt L'Hospital's rule

1 pt $x = \frac{1}{h}$ $\quad f'(0) = \lim_{x \to \pm\infty} \dfrac{x}{e^{x^2}}$

1 pt $= \lim_{x \to \pm\infty} \dfrac{1}{2x\, e^{x^2}} = 0$

b) $f'(x)$ differentiable at $x = 0$?

1 pt does $\lim_{h \to 0} \dfrac{f'(h) - f'(0)}{h}$ exists ?

does $\lim_{h \to 0} \dfrac{2h^{-3} e^{-1/h^2}}{h}$ exist ?

h

1pt does $2\lim\limits_{h\to 0} \dfrac{e^{-1/h^2}}{h^4}$ exist?

$x^2 = \frac{1}{h^2}$

1pt does $2\lim\limits_{x\to\infty} \dfrac{x^2}{e^x}$ exists?

l'Hospital

1pt $2\lim\limits_{x\to\infty} \dfrac{x^2}{e^x} = 2\lim\limits_{x\to\infty} \dfrac{2x}{e^x} = 4\lim\limits_{x\to\infty} \dfrac{1}{e^x} = 0$

1pt f' is differentiable at $x=0$

Ex 6 a) Evaluate $\lim\limits_{x\to 0} (1-2x)^{1/x}$

1pt $(1-2x)^{1/x} = e^{\frac{1}{x}\ln(1-2x)}$

1pt $\lim\limits_{x\to 0}(1-2x)^{1/x} = e^{\lim\limits_{x\to 0} \frac{1}{x}\ln(1-2x)}$

$\lim\limits_{x\to 0} \frac{1}{x}\ln(1-2x)$ indeterminate form

1pt l'Hopital

1pt $\lim\limits_{x\to 0} \frac{1}{x}\ln(1-2x) = \lim\limits_{x\to 0} \dfrac{\frac{1}{1-2x}(-2)}{1} =$

1pt $\lim\limits_{x\to 0} \dfrac{-2}{1-2x} = -2$

-2

Hence $\lim_{x \to 0} (1-2x)^{1/x} = e^{-2}$

b)

1pt $F(x) = \int_0^x f(t)\,dt \Rightarrow F'(x) = f(x)$

(i)

$$\int_x^0 f(t)\,dt = e^{x^2}$$

$$- \int_0^x f(t)\,dt = e^{x^2}$$

1pt differentiate $-f(x) = 2x\,e^{x^2}$

1pt hence $f(0) = 0$

1pt $\int_0^u$

differentiate wrt u $\quad f(u) = e^u$

1pt $f(0) = 1$

Ex 7 a) $\int \sqrt{1+x^2}\; x^5\,dx =$

1pt $\frac{1}{2}\int \sqrt{1+x^2}\; x^4 \frac{d}{dx}(1+x^2)\,dx =$

1pt $[\,1+x^2 = u \qquad x^2 = u-1\,]$

$\frac{1}{2}\int \sqrt{u}\,(u-1)^2\,du =$

1pt $\frac{1}{2}\int u^{1/2}(u^2-2u+1)\,du =$

1pt $\frac{1}{2}\int u^{1/2}\,(u^2-2u+1)\,du =$

1pt $\frac{1}{2}\left(\frac{2}{7}u^{7/2} - 2\,\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C =$

1pt $\frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C$

b) $\int_1^e (\ln x)^2\,dx$

1pt idea integration by parts

1pt $\int \ln x\,dx = x\ln x - x \quad (+c)$

$\int_1^e \ln x \ln x\,dx =$

1pt $\ln x\,[x\ln x - x]\Big|_1^e - \int_1^e \frac{1}{x}(x\ln x - x)\,dx$

1pt $= 0 - \int_1^e \ln x\,dx + \int_1^e 1\,dx$

$= -(x\ln x - x)\Big|_1^e + x\Big|_1^e$

1pt $= e - 2$

Alternative

1pt $\int_1^e (\ln x)^2 \cdot 1\,dx =$

1pt $(\ln x)^2 \cdot x\Big|_1^e - \int_1^e 2\ln x \cdot \frac{1}{x} \cdot x\,dx =$

1pt $e - 2\int_1^e \ln x\,dx =$

1 pt $\quad e - 2\,(x\ln x - x)\Big|_1^e =$

1 pt $\quad e - 2\,(e - e + 1) = e - 2$

Ex 8 $\quad xy' - 2y = x^2$

1 pt $\quad y' - \frac{2}{x}y = x$

Integrating factor

1 pt $\quad (Iy)' = I\left(y' - \frac{2}{x}y\right)$

$\Leftarrow \quad I' = I\left(-\frac{2}{x}\right)$

1 pt

3 pt $\quad I \overset{!}{=} e^{\int -\frac{2}{x}\,dx}$

$e^{-2\ln x}$

1 pt

1 pt $\quad = \frac{1}{x^2}$ (integration const ok)

2 pt $\quad (Iy)' = Ix = \frac{1}{x}$

1 pt $\quad Iy = \ln|x| + c \qquad c \in \mathbb{R}.$

1 pt $\quad y(x) = x^2\ln|x| + x^2 c$